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Uniform Approximation with Positive Linear Operators Generated by Binomial Expansions

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Uniform approximation of functions of a real or a complex variable by a class of linear operators generated by certain power series is studied.

1. INTRODUCTION

Let $\phi(y)$, ϕ^{-1} be analytic for $|y| < r \leqslant \infty$, with

$$\phi(y) = \sum_{k=0}^{\infty} a_k y^k$$

and $a_0 > 0$, $a_1 > 0$; $a_k \geqslant 0$, $k = 2, 3, \ldots$. There exist [3] $b \in (0, \infty]$, a domain D containing the origin, and a unique function f such that $f(0) = 0$, f is analytic on D, $f(x) > 0$ for $0 < x < b$, and

$$\frac{f(z)\,\phi'(f(z))}{\phi(f(z))} = z, \qquad z \in D. \tag{1.1}$$

For $n = 1, 2, \ldots$ and $|y| < r$, let

$$\phi_n(y) = [\phi(y)]^n = \sum_{k=0}^{\infty} a_{nk} y^k. \tag{1.2}$$

Define the linear operator

$$L_n(h; z) = [\phi_n(f(z))]^{-1} \sum_{k=0}^{\infty} a_{nk} (f(z))^k h\left(\frac{k}{n}\right) \tag{1.3}$$

for those h, z for which the right-hand side of (1.3) exists. For example, if h is bounded on the positive axis and $|z|$ is sufficiently small, then (1.3) exists

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and is also a positive operator on some interval $[0, a]$. This method specializes one introduced by Pethe [3], who obtained uniform approximation of bounded functions. The author [5] studied the order of approximation of bounded functions with linear combinations of the special case of (1.3) defined below.

Assume $\phi(0)=1$ and $\phi'(y)=[\phi(y)]^{1+\alpha}$ with $\alpha=-1/m$, $m=1, 2, ...,$ or $a \geqslant 0$. If $\alpha=-1/m$ then $\phi(y)=(1+y/m)^m$, $f(z)=z(1-z/m)^{-1}$, and (1.3) is the generalized Bernstein polynomial

$$L_n(h; z)=\sum_{k=0}^{mn}\binom{mn}{k}\left(\frac{z}{m}\right)^k\left(1-\frac{z}{m}\right)^{mn-k} h\left(\frac{k}{n}\right). \tag{1.4}$$

If $\alpha=0$ then $\phi(y)=e^y$, $f(z)=z$, and (1.3) is just the Szász operator. If $\alpha>0$ then $\phi(y)=(1-\alpha y)^{-\alpha^{-1}}$, $f(z)=z(1+\alpha z)^{-1}$, and (1.3) is a generalized Baskakov method. For $\alpha=1/m$, $m=1, 2, ...,$ we have

$$L_n(h; z)=\left(1+\frac{z}{m}\right)^{-mn}\sum_{k=0}^{\infty}\binom{mn+k-1}{k}\left(\frac{z}{m+z}\right)^k h\left(\frac{k}{n}\right). \tag{1.5}$$

The Baskakov operator is obtained when $\alpha=1$.

Becker [1] has discussed weighted global approximation for the Szász and Baskakov operators. In Section 2 we derive his direct result for (1.3) in the case $\alpha>0$ and also approximate functions having exponential growth on $[0, \infty]$. Section 3 contains results on the approximation of analytic functions.

2. Approximation on the Real Line

In this section L_n denotes (1.3) with $\phi(y)=(1-\alpha y)^{-\alpha^{-i}}$ and $\alpha>0$.

LEMMA 2.1. *For $x \geqslant 0$ and $n=1, 2, ...,$*

$$L_n(1; x)=1; \tag{2.1}$$

$$L_n(t; x)=x; \tag{2.2}$$

$$L_n((t-x)^2; x)=\frac{\alpha x^2+x}{n}. \tag{2.3}$$

Proof. The results follow easily from (1.1), (1.2), (1.3), and $\phi'(y)=[\phi(y)]^{1+\alpha}$.

LEMMA 2.2. *For $x \geqslant 0$, $n = 1, 2, \ldots$, and $r = 2, 3, \ldots$,*

$$L_n(t^r; x) = \sum_{l=1}^{r} \sigma_r^l \frac{n(n+\alpha)\cdots(n+(l-1)\alpha)}{n^r} x^l, \tag{2.4}$$

where σ_r^l are Stirling numbers of the second kind [2].

Proof. Let $y = f(x) = x(1+\alpha x)^{-1}$, $x \geqslant 0$, in (1.1). Using (1.2),

$$\frac{y\phi_n'(y)}{\phi_n(y)} = n\frac{y\phi'(y)}{\phi(y)} = nx.$$

Assume

$$\frac{y^l\phi_n^{(l)}(y)}{\phi_n(y)} = n(n+\alpha)\cdots(n+(l-1)\alpha)\,x^l.$$

Using $\phi'(y) = [\phi(y)]^{1+\alpha}$ we obtain

$$\frac{y^{l+1}\phi_n^{(l+1)}(y)}{\phi_n(y)} = n(n+\alpha)\cdots(n+(l-1)\alpha)(n+l\alpha)\,x^{l+1}$$

and hence, for $l = 1, 2, \ldots$, $x \geqslant 0$, $y = f(x)$,

$$\frac{y^l\phi_n^{(l)}(y)}{\phi_n(y)} = n(n+\alpha)\cdots(n+(l-1)\alpha)\,x^l.$$

If $x \geqslant 0$ then $0 \leqslant y < 1/\alpha$ and

$$\begin{aligned} L_n(t^r; x) &= [\phi_n(y)]^{-1} \sum_{k=0}^{\infty} a_{nk} y^k \left(\frac{k}{n}\right)^r \\ &= [\phi_n(y)]^{-1} \sum_{k=0}^{\infty} a_{nk} y^k \sum_{l=1}^{r} \sigma_r^l \frac{k(k-1)\cdots(k-l+1)}{n^r} \\ &= \sum_{l=1}^{r} \sigma_r^l \frac{y^l\phi_n^{(l)}(y)}{n^r\phi_n(y)} \\ &= \sum_{l=1}^{r} \sigma_r^l \frac{n(n+\alpha)\cdots(n+(l-1)\alpha)}{n^r} x^l. \end{aligned}$$

Lemma 2.2 shows that the numbers $a_{r,j}$ in Lemma 3 of [1] are Stirling numbers, since (2.4) is also valid for the Szász operator ($\alpha = 0$ and $f(x) = x$).

Define weights $w_0(x) = 1$, $w_N(x) = (1 + x^N)^{-1}$, $x \geqslant 0$, $N = 1, 2, \ldots$, and space $C_N = \{h: h \in C[0, \infty)$ and $w_N h$ is uniformly continuous and bounded on $[0, \infty)\}$. For $h \in C_N$ define [1]

$$\|h\|_N = \sup_{x \geq 0} w_N(x)\, |h(x)|,$$

$$\Delta_\lambda^2\, h(x) = h(x+2\lambda) - 2h(x+\lambda) + h(x), \qquad \lambda > 0,$$

and

$$\omega_N^2(h, \delta) = \sup_{0 < \lambda \leq \delta} \|\Delta_\lambda^2 h\|_N.$$

In the sequel the finite constants $M_{N,\alpha}$ may have different values at each occurrence. The next result is an easy consequence of (2.4).

LEMMA 2.3. *For* $N = 0, 1, \ldots, x \geq 0$ $n = 1, 2, \ldots,$

$$w_N(x)\, L_n(1 + t^N; x) \leq M_{N,\alpha}. \tag{2.5}$$

LEMMA 2.4. *For* $N = 0, 1, \ldots, h \in C_N$, *and* $n = 1, 2, \ldots,$

$$\|L_n(h)\|_N \leq M_{N,\alpha}\, \|h\|_N. \tag{2.6}$$

Proof. If $x \geq 0$,

$$w_N(x)\, |L_n(h; x)| \leq w_N(x) \|h\|_N L_n(1 + t^N; x)$$

and (2.6) follows from (2.5).

LEMMA 2.5. *For* $N = 0, 1, \ldots, x \geq 0$, *and* $n = 1, 2, \ldots,$

$$w_N(x)\, L_n((t-x)^2(1+t^N); x) \leq M_{N,\alpha} \left(\frac{\alpha x^2 + x}{n} \right). \tag{2.7}$$

Proof. The result is trivial for $N = 0$. For $N \geq 1$ and $x \geq 0$, using (2.4)

$$0 \leq L_n((t-x)^2 t^N; x) = L_n(t^{N+2}; x) - 2x L_n(t^{N+1}; x) + x^2 L_n(t^N; x)$$

$$= \frac{n \cdots (n + (N+1)\,\alpha)}{n^{N+2}} x^{N+2} + \sigma_{N+2}^{N+1} \frac{n \cdots (n + N\alpha)}{n^{N+2}} x^{N+1} \cdots$$

$$+ \frac{x}{n^{N+1}} - 2 \left[\frac{n \cdots (n + (N)\,\alpha)}{n^{N+1}} x^{N+2} \right.$$

$$\left. + \sigma_{N+1}^{N} \frac{n \cdots (n + (N-1)\,\alpha)}{n^{N+1}} x^{N+1} + \cdots + \frac{x^2}{n^N} \right]$$

$$+\frac{n\cdots(n+(N-1)\alpha)}{n^N}x^{N+2}$$

$$+\sigma_N^{N-1}\frac{n\cdots(n+(N-2)\alpha)}{n^N}x^{N+1}+\cdots+\frac{x^3}{n^{N-1}}$$

$$=\left(1+\frac{\alpha}{n}\right)\left(1+\frac{2\alpha}{n}\right)\cdots\left(1+\frac{(N-1)\alpha}{n}\right)\left[\left(1+\frac{N\alpha}{n}\right)\left(1+\frac{(N+1)\alpha}{n}\right)\right.$$

$$\left.-2\left(1+\frac{N\alpha}{n}\right)+1\right]x^{N+2}+\cdots+\frac{x}{n^{N+1}}=\frac{\alpha x^2}{n}A_{N,n,\alpha}(x)+\frac{x}{n^{N+1}},$$

where $A_{N,n,\alpha}(x)$ is a polynomial in x of degree N with coefficients that are bounded in n. Estimate (2.7) follows from (2.3) and the above.

We can now state the direct result. The proof is exactly the same as [1, Theorem 8], using (2.1), (2.2), (2.3), (2.6), and (2.7).

THEOREM 2.6. *For $N=0, 1, ..., h\in C_N$, $x\geqslant 0$, and $n=1, 2, ...$,*

$$w_N(x)|L_n(h;x)-h(x)|\leqslant M_{N,\alpha}\,\omega_N^2\left(h,\left(\frac{\theta(x)}{n}\right)^{1/2}\right), \tag{2.8}$$

where $\theta(x)=\alpha x^2+x$.

Since Becker [1] has shown (2.8) for the Szász operator ($\alpha=0$), the Szász operator provides the best weighted global approximation for the class of methods generated by the analytic function ϕ with $\phi(0)=1$ and $\phi'(y)=[\phi(y)]^{1+\alpha}$, $\alpha\geqslant 0$. Also, (2.8) implies uniform convergence on $[0, a]$ for functions, h, having polynomial growth on $[0, \infty)$. Our next theorem yields uniform convergence for functions with exponential growth on the positive axis.

THEOREM 2.7. *If h is continuous on $[0, a]$, continuous from the right at a, and, for some finite number A, $|h(x)|\leqslant e^{Ax}$, $x\geqslant 0$, then*

$$\lim_{n\to\infty} L_n(h;x)=h(x) \tag{2.9}$$

uniformly on $[0, a]$.

Proof. Choose $S>A/\ln(1/\alpha a+1)$ and let $n\geqslant S$, $0\leqslant x\leqslant a$. Then

$$0\leqslant\frac{e^{A/n}x}{1+\alpha x}<\frac{1}{\alpha}$$

and

$$L_n(e^{At}; x) = \left[\frac{\phi(e^{A/n}\, x/1+\alpha x)}{\phi(x/1+\alpha x)}\right]^n = [1-(e^{A/n}-1)\,\alpha x]^{-n/\alpha}.$$

Using L'Hopital's rule,

$$\lim_{n\to\infty} L_n(e^{At}; x) = e^{Ax} \tag{2.10}$$

uniformly on $[0, a]$.

Let $0 \leqslant x \leqslant a$, $\varepsilon > 0$, and choose $\delta > 0$ such that $|h(t) - h(x)| < \varepsilon$ if $|t-x| < \delta$, $0 \leqslant x \leqslant a$, and $0 \leqslant t < a + \delta$. Let

$$c_{nk}(x) = \frac{a_{nk}(f(x))^k}{\phi_n(f(x))},$$

where $f(x) = x(1+\alpha x)^{-1}$ and $n > 2A/\ln(1/\alpha a + 1)$. Then

$$\begin{aligned}|L_n(h; x) - h(x)| &\leqslant \sum_{|k/n - x| < \delta} c_{nk}(x) \left| h\left(\frac{k}{n}\right) - h(x)\right| \\ &+ \sum_{|k/n - x| \geqslant \delta} c_{nk}(x) \left| h\left(\frac{k}{n}\right) - h(x)\right| \\ &< \varepsilon + \frac{e^{Ax}}{\delta^2} \sum_{k=0}^{\infty} \left(\frac{k}{n} - x\right)^2 c_{nk}(x) + \frac{1}{\delta} \sum_{k=0}^{\infty} \left|\frac{k}{n} - x\right| c_{nk}(x)\, e^{Ak/n} \\ &\leqslant \varepsilon + \frac{e^{Ax}}{\delta^2} L_n((t-x)^2; x) + \frac{1}{\delta} [L_n((t-x)^2; x)\, L_n(e^{2At}; x)]^{1/2}\end{aligned}$$

and (2.9) follows from (2.3) and (2.10).

If $h \in C[0, \infty)$ and, for some finite A,

$$\overline{\lim_{x\to\infty}} \frac{|h(x)|}{e^{Ax}} < \infty,$$

the proof can easily be modified to yield (2.9). We cannot allow h to grow any faster. For example, let $h(x) = e^{x^{1+\varepsilon}}$, $\varepsilon > 0$. For $n \geqslant 1$ and $x > 0$,

$$\begin{aligned}L_n(h; x) &\geqslant (1+\alpha x)^{-n/\alpha} \sum_{k=1}^{\infty} \frac{n(n+\alpha)\cdots(n+(k-1)\,\alpha)}{k!} \left(\frac{x}{1+\alpha x}\right)^k (e^{k/n})^{1+\varepsilon} \\ &\geqslant (1+\alpha x)^{-n/\alpha} \sum_{k=1}^{\infty} \left(\frac{nx}{1+\alpha x}\right) \frac{k(e^{k/n})^{1+\varepsilon}}{k!}\end{aligned}$$

and the last series is divergent. Theorem 2.7 is well known for the Szász operator (see [4]).

3. Approximation in the Complex Plane

Our final results deal with the approximation of analytic functions. In the first theorem, L_n is the generalized Baskakov method of Section 2, while L_n denotes the generalized Bernstein polynomial (1.4) in Lemma 3.2 and Theorem 3.3.

THEOREM 3.1. *If* $\alpha>0$, *h is entire, and for some finite number A*, $|h(x)|\leqslant e^{Ax}$, $x\geqslant 0$, *then*

$$\lim_{n\to\infty} L_n(h; z)=h(z) \tag{3.1}$$

uniformly on compact subsets of $\operatorname{Re} z>-1/2\alpha$.

Proof. Let

$$h(z)=\sum_{v=0}^{\infty} \alpha_v z^v, \qquad |z|<\infty.$$

Without loss of generality, we may assume $\alpha_v\geqslant 0$, $v=0, 1, ...,$ since we can write $h=h_1-h_2+i(h_3-h_4)$ where the h_i's have nonnegative Taylor coefficients. Let S be a compact subset of $\operatorname{Re} z>-1/2\alpha$. Since $f(z)=z(1+\alpha z)^{-1}$ is analytic on $\operatorname{Re} z>-1/2\alpha$ and maps that set into $|y|<1/\alpha$, $f(S)$ is a compact subset of $|y|<1/\alpha$. Thus there exists λ such that $z\in S$ implies $|z/(1+\alpha z)|\leqslant 1/\lambda<1/\alpha$. Hence $L_n(t^v; z)$ is analytic on $\operatorname{Re} z>-1/2\alpha$ for $v=0, 1, 2, ...$ Let

$$\Omega_n=\left\{z: \left|\frac{ze^{A/n}}{1+\alpha z}\right|<\frac{1}{\alpha}\right\}=\{z: z=x+iy \text{ and } (x-c_n)^2+y^2<r_n^2\},$$

where

$$c_n=\frac{1}{\alpha(e^{2A/n}-1)}, \qquad r_n=\frac{e^{A/n}}{\alpha(e^{2A/n}-1)}.$$

Since $|h(x)|\leqslant e^{Ax}$ for $x\geqslant 0$, an argument similar to the above shows $L_n(h; z)$ is analytic on Ω_n for each $n=1, 2, ...$ Next $\Omega_n\subset\Omega_{n+1}\subset\cdots\subset \operatorname{Re} z>-1/2\alpha$. Choose N such that $\alpha/\lambda<1/e^{A/N}<1$ and it follows that $S\subset\Omega_n$ for $n\geqslant N$. Theorem 2.7 gives

$$\lim_{n\to\infty} L_n(h, x)=h(x), \qquad 0\leqslant x\leqslant c_N+r_N.$$

Since $L_n(t^v; z)$ is analytic for Re $z > -1/2\alpha$, using (2.4) we see that (2.4) holds for Re $z > -1/2\alpha$. If $z \in \Omega_N$ then

$$\sum_{v=0}^{\infty} \alpha_v |L_n(t^v; z)| \leqslant \sum_{v=0}^{\infty} \alpha_v L_n(t^v; |z|) \leqslant L_n(h; c_N + r_N) < \infty.$$

Hence

$$\sum_{v=0}^{\infty} \alpha_v L_n(t^v; z)$$

is analytic for $z \in \Omega_N$, $n = 1, 2, \ldots$. Since

$$L_n(h; x) = \sum_{v=0}^{\infty} \alpha_v L_n(t^v; x)$$

for $0 \leqslant x \leqslant c_N + r_N$, we have

$$L_n(h; z) = \sum_{v=0}^{\infty} \alpha_v L_n(t^v; z)$$

for $z \in \Omega_N$ and $n \geqslant N$. Finally, $\{L_n(h;\ )\}$, $n \geqslant N$, is a uniformly bounded sequence on compact subsets of Ω_N. By Vitali's convergence theorem

$$\lim_{n \to \infty} L_n(h; z) = h(z)$$

uniformly on compact subsets of Ω_N and (3.1) follows.

Analogous to the remark following Theorem 2.7, the proof of Theorem 3.1 can be modified to obtain conclusion (3.1) if h is entire and there exists a finite number A such that

$$\overline{\lim_{x \to \infty}} \frac{|h(x)|}{e^{Ax}} < \infty.$$

In particular, Theorem 3.1 is valid for entire functions of exponential type. For the Szász operator, uniform convergence is obtained on compact subsets of the finite plane [4].

In the sequel let $\phi(y) = (1 + y/m)^m, f(z) = z(1 - z/m)^{-1}$, $m = 1, 2, \ldots$, and

$$c_{nk}(z) = \frac{a_{nk}(f(z))^k}{[\phi(f(z))]^n}.$$

LEMMA 3.2. *For* $0 < x < m$, $k = 0, 1, \ldots$, *and* $n = 1, 2, \ldots$,

$$\frac{1}{n}\left(x - \frac{x^2}{m}\right) c'_{nk}(x) = \left(\frac{k}{n} - x\right) c_{nk}(x). \tag{3.2}$$

Proof. Let $0 < x < m$. Using $\phi'(y) = [\phi(y)]^{1-1/m}$,

$$c'_{nk}(x) = a_{nk}\left\{\frac{k(f(x))^{k-1}f'(x)}{[\phi(f(x))]^n} - \frac{(f(x))^k f'(x)\,n}{[\phi(f(x))]^{n+1/m}}\right\}$$

$$= nf'(x)[\phi(f(x))]^{-1/m}\left\{\frac{ka_{nk}(f(x))^{k-1}}{n[\phi(f(x))]^{n-1/m}} - \frac{a_{nk}(f(x))^k}{[\phi(f(x))]^n}\right\}.$$

But

$$x = \frac{f(x)\,\phi'(f(x))}{\phi(f(x))} = f(x)[\phi(f(x))]^{-1/m}$$

and hence

$$1 = \frac{f'(x)[\phi(f(x))]^{-1/m}f(x)}{f(x)} - \frac{f'(x)\{f(x)[\phi(f(x))]^{-1/m}\}^2}{mf(x)}$$

$$= \frac{f'(x)}{f(x)}\left(x - \frac{x^2}{m}\right).$$

Therefore

$$\frac{1}{n}\left(x - \frac{x^2}{m}\right)c'_{nk}(x) = \frac{ka_{nk}(f(x))^k f'(x)}{n[\phi(f(x))]^n f(x)}\left(x - \frac{x^2}{m}\right)$$

$$- \frac{a_{nk}(f(x))^k f'(x)}{[\phi(f(x))]^n}\left(x - \frac{x^2}{m}\right)[\phi(f(x))]^{-1/m}$$

$$= \frac{a_{nk}(f(x))^k}{[\phi(f(x))]^n}\left(\frac{k}{n} - x\right)$$

$$= c_{nk}(x)\left(\frac{k}{n} - x\right).$$

THEOREM 3.3. *If m is a positive integer,*

$$h(z) = \sum_{v=0}^{\infty} \alpha_v z^v, \qquad |z| \leqslant m,$$

with

$$\sum_{v=0}^{\infty} |\alpha_v|\, m^v < \infty,$$

and $L_n(h; z)$ *is the generalized Bernstein polynomial* (1.4), *then*

$$\lim_{n \to \infty} L_n(h; z) = h(z) \tag{3.3}$$

uniformly on $|z| \leqslant m$.

Proof. As in the proof of Theorem 3.1 we may assume $\alpha_v \geqslant 0$, $v = 0, 1, \ldots$. From [3] we have

$$\lim_{n \to \infty} L_n(h; x) = h(x), \qquad 0 \leqslant x \leqslant m.$$

The proof of Lemma 2.2 with $\alpha = -1/m$, $\phi(y) = (1 + y/m)^m$, and $y = f(x) = x(1 - x/m)^{-1}$ shows that (2.4) holds for $0 \leqslant x < m$. Therefore

$$L_n(t^v; z) = \sum_{l=1}^{v} \sigma_v^l \frac{n(n-1/m) \cdots (n-(l-1)/m)}{n^v} z^l \tag{3.4}$$

for all complex z. When $r > mn$ the coefficients of z^l vanish for $l > mn$. From (3.4) and

$$L_n(h; x) = \sum_{v=0}^{\infty} \alpha_v L_n(t^v; x), \qquad 0 \leqslant x \leqslant m,$$

we see that

$$\sum_{v=0}^{\infty} \alpha_v L_n(t^v; z)$$

converges uniformly on $|z| \leqslant m$. Hence

$$L_n(h; z) = \sum_{v=0}^{\infty} \alpha_v L_n(t^v; z), \qquad |z| < m.$$

Just as in the proof of Theorem 3.1,

$$\lim_{n \to \infty} L_n(h; z) = h(z)$$

uniformly on compact subsets of $|z| < m$. In particular, we have uniform convergence on each disk $|z| \leqslant p < m$. Using (3.4),

$$|L_n'(h; z)| \leqslant L_n'(h; p) \leqslant L_n'(h; m)$$

for $|z| \leqslant p < m$. By continuity

$$|L_n'(h; z)| \leqslant L_n'(h; p)$$

for $|z| \leqslant p \leqslant m$. Next, for any $|z| \leqslant m$, $p \leqslant |z| \leqslant m$, $z = te^{i\eta}$, we have

$$|L_n(h; z) - L_n(h; pe^{i\eta})| \leqslant \int_p^t |L_n'(h; xe^{i\eta})| \, dx$$
$$\leqslant L_n(h; t) - L_n(h; p) \leqslant (t-p) \, L_n'(h; m).$$

Therefore the functions $L_n(h; z)$ will be equicontinuous in $|z| \leqslant m$ if the sequence $\{L_n'(h; m)\}$ is bounded. From (2.1), (2.2), which are true for (1.4), and (3.2)

$$\begin{aligned} 0 \leqslant L_n'(h; x) &= \left(\frac{n}{x - x^2/m}\right) L_n((t-x) \, h(t); x) \\ &= \left(\frac{n}{x - x^2/m}\right) L_n((t-x)(h(x) + h'(\zeta)(t-x)); x) \\ &= \left(\frac{n}{x - x^2/m}\right) L_n(h'(\zeta)(t-x)^2; x) \\ &\leqslant \left(\frac{n}{x - x^2/m}\right) L_n((t-x)^2; x) \, h'(m) = h'(m) \end{aligned}$$

for $0 < x < m$. By continuity,

$$0 \leqslant L_n'(h; m) \leqslant h'(m).$$

Finally, since the $L_n(h; z)$ converge uniformly to $h(z)$ on each disk $|z| \leqslant p < m$, and are equicontinuous on $|z| \leqslant m$, they converge uniformly on $|z| \leqslant m$ and (3.3) is proved.

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